Lattice QCD in strong magnetic backgrounds

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● Strong interactions are described by QCD, the theory of quarks and gluons.
● Quarks are also subject to electroweak interactions, which in general induce small corrections to strong interaction dynamics, but exceptions are expected in presence of strong e.m. backgrounds, a situations which is relevant to many contexts:

- Large magnetic fields ($B \sim 10^{10}$ Tesla) are expected in a class of neutron stars known as magnetars (Duncan-Thompson, 1992).
- Large magnetic fields ($B \sim 10^{16}$ Tesla, $\sqrt{|e|B} \sim 1.5$ GeV), may have been produced at the cosmological electroweak phase transition (Vachaspati, 1991).

- in non-central heavy ion collisions, largest magnetic fields ever created in a laboratory ($B$ up to $10^{15}$ Tesla at LHC) with a possible rich associated phenomenology: chiral magnetic effect (Vilenkin, 1980; Kharzeev, Fukushima, McLerran and Warringa, 2008).
E.m. fields affect quarks directly and gluons only at the 1-loop level. However non-perturbative effects can be non-trivial in the gluon sector as well. Various model computations predict a rich phenomenology:

- Effects on the QCD vacuum structure:
  - chiral symmetry breaking? Quite natural (Magnetic catalysis of $\chi^{SB}$)
  - confinement? Less obvious (see later)

- Effects on the QCD phase diagram:

- Equation of state: is strongly interacting matter paramagnetic or diamagnetic?

LQCD is the ideal tool for a non-perturbative investigation of such issues. QCD+QED studies of the e.m. properties of hadrons go back to the early days of LQCD


Recent years have seen an increasing activity on the subject.
LQCD in electromagnetic background fields

An e.m. background field $a_\mu$ modifies the continuum covariant derivative as follows:

$$D_\mu = \partial_\mu + i g A_\mu^a T^a \rightarrow \partial_\mu + i g A_\mu^a T^a + i q a_\mu$$

in the lattice formulation, the simplest symmetric discretization is

$$D_\mu \psi \rightarrow \frac{1}{2a} \left( U_\mu(n) u_\mu(n) \psi(n + \hat{\mu}) - U_\mu^\dagger(n - \hat{\mu}) u_\mu^*(n - \hat{\mu}) \psi(n - \hat{\mu}) \right)$$

$U_\mu \in SU(3)$

$u_\mu \simeq \exp(i q a_\mu(n)) \in U(1)$ depends on the quark charge $q$. 
The thermal partition function of QCD is written as usual in terms of an euclidean path integral, with

\[ T = \frac{1}{\tau} = \frac{1}{N_t a(\beta, m)} \]

where \( \tau \) is the extension of the compactified time

\[
Z = \text{Tr} \left( e^{-\frac{H_T}{T}} \right) \Rightarrow \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\left(S_G[U] + \bar{\psi}M[U,u]\psi\right)} = \int \mathcal{D}U e^{-S_G[U]} \det M[U, u]
\]

where \( M \) is the fermion matrix

- \( u \) fields affect gluon fields through the quark determinant and are not dynamical in the following (no integration): **quenched QED approach.**

- By loop expansion of the determinant (loop \( \in U(3) \)) or by \( \gamma_5 M^\dagger \gamma_5 = M \)

\[
\det M[U, u] > 0 \quad \Rightarrow \quad \text{MC simulations are feasible} \quad \text{(with a caveat for electric fields)}
\]
Some limitations and constraints

- **Field quantization on compact manifolds:**
  
  - To minimize finite size effects, one usually works on a compact manifold, like a torus (periodic b.c.).
  
  Like for magnetic monopoles, consistency conditions for the gauge phases picked up by charged particles impose a quantized field flux through each closed surface (‘t Hooft, 1979).

  - e.g. for $\vec{B} = B \hat{z}$ on a torus populated by particles of charge $q$:

    $qB = \frac{2\pi b}{L_x L_y a^2}$

    where $b$ is an integer
Consider an \( l_x \times l_y \) torus and a realization of \( \vec{B} = B \hat{z} \): \( A_x = 0, A_y = B x \)
- this is discontinuous at \( x = 0 \): that can be cured by adding \( A(x) = -\delta(x) B l_x \)
- but then \( A(x) \) is discontinuous in \( y = 0 \), and that cannot be cured any more

Particles looping around the origin will take a wrong \(-q B l_x l_y\) additional phase
we are left with a uniform field plus a Dirac string, which is invisible only for quantized fields
The lattice $U(1)$ links corresponding to the choice above are the following:

$$u_y(B, q)(n) = e^{i\alpha^2 qB n_x}; \quad u_\mu(B, q)(n) = 1 \text{ for } \mu = x, z, t; \quad u_x(B, q)(n)|_{n_x = L_x} = e^{-i\alpha^2 qL_x B n_y}$$

they corresponds to a uniform field plus a Dirac string in the origin of each $xy$ surface, which is invisible for integer $b$.

The Dirac string can actually be moved anywhere on the torus, for integer $b$ this is done by a simple gauge transformation.

- **UV limitations from discretization:**

  the plaquette sets the minimum explorable flux on the lattice, which is defined up to a $2\pi$ phase, thus fixing a sort of first Brillouin zone:

  $$-\frac{\pi}{\alpha^2} < qB < \frac{\pi}{\alpha^2}$$

- **As for electric fields:**

  Real electric fields in Euclidean correspond to purely imaginary fields in Minkowski. If you want real electric fields in Minkowski, then you get a sign problem.
We can only simulate constant magnetic fields (better if uniform) and compute equilibrium properties. That may be different from experimental conditions and probes.

Estimate of $eB$ time evolution @ RHIC for $Au - Au$ collisions for two values of $\sqrt{s_{NN}}$.

As the collision energy increases the magnetic field increases, but it gets more shrinked in time.

[Skokov, Illarionov and Toneev, '09]

Accurate predictions about the magnetic field evolution requires knowledge of the medium conductivity.
Magnetic catalysis ($T = 0$)

The enhancement of chiral symmetry breaking in presence of a magnetic field has been predicted by a variety of model computations. A useful quantity is:

$$r(B) \equiv \frac{\langle \bar{\psi}\psi \rangle (B) - \langle \bar{\psi}\psi \rangle (B = 0)}{\langle \bar{\psi}\psi \rangle (B = 0)}.$$

The general expectation is that $\langle \bar{\psi}\psi \rangle$ be even in $B$, hence $r \propto B^2$, for small enough $B$, if the theory is analytic at $B = 0$. This is not true for massless quarks. From $\chi$PT:

- for $m_\pi = 0$ and $|e|B \ll \Lambda_{QCD}^2$ (Shushpanov and Smilga, 1997)

$$r(B) = \frac{\log(2) |e||B|}{16\pi^2 F_\pi^2}.$$

- for $m_\pi \neq 0$ and $|e|B \ll m_\pi^2$ (Cohen, McGady and Werbos, 2007)

$$r(B) \approx \frac{(|e|B)^2}{96\pi^2 F_\pi^2 m_\pi^2}.$$
One of the first lattice studies about the enhancement of chiral symmetry breaking induced has been done by the ITEP group (Buividovich, Chernodub, Luschevskaya and Polikarpov, Phys. Lett. B 682, 484 (2010) and arXiv:1011.3795)

- pure gauge investigation with $N_c = 2$. The magnetic field only enters the (overlap) Dirac operator for the computation of $\langle \bar{\psi} \psi \rangle$. 
Later studies have considered the effects on the gauge field distribution by including dynamical quark contributions (D’E. and Negro, 2011, Ilgenfritz et al. 2012, Bali et al., 2012).

In particular one can try to disentangle the contribution from the modified operator (”valence”) from that coming from the modified gauge field distribution (”dynamical” or “sea” contribution) (M. D’E and Negro, arXiv:1103.2080)

\[ \langle \bar{\psi} \psi \rangle^{val}_{u/d}(B) \equiv \int \mathcal{D}U \mathcal{P}[m, U, 0] \text{Tr} (M^{-1}[m, B, q_{u/d}]) \]

\[ \langle \bar{\psi} \psi \rangle^{dyn}_{u/d}(B) \equiv \int \mathcal{D}U \mathcal{P}[m, U, B] \text{Tr} (M^{-1}[m, 0, q_{u/d}]) \]

\( M \) here is the standard staggered matrix and \( \mathcal{P} \propto e^{-S_G[U]} \det M^{\frac{1}{2}}[U, q_u] \det M^{\frac{1}{2}}[U, q_d] \), corresponding to \( N_f = 2 \)

The separation is expected to be well defined, at least for \( B \) small enough.
From arXiv:1103.2080, $m_\pi \sim 200$ MeV, unimproved staggered fermions, $16^4$ lattice, $a \sim 0.3$ fm

**Left:** relative increment $r(B)$ for $u$ and $d$ quarks and their average.

**Right:** "valence" and "dynamical" contribution to the average condensate.

- $r(B)$ quadratic for small $B$. $\chi$PT prediction works well in a wider range
- Separation into "valence" and "dynamical" contribution well defined over a wide range
- The modification of gauge field distribution accounts for about 40% of the total amount of magnetic catalysis
Later study with improved action: G. Bali et al., arXiv:1206.4205. $N_f = 2 + 1$, stout smeared staggered fermions, physical quark masses

**Left:** relative increment $r(B)$ for the average $u$-$d$ condensate at various spacings and continuum extrapolation

**Right:** Comparison with $\chi$PT and PNJL model predictions

- UV cutoff effects seems negligible, continuum extrapolation quite smooth
- Nice agreement with PNJL model prediction
Magnetic field effects on the QCD transition

There are several aspects that one would like to investigate:

- How $T_c$ changes as a function of $B$?
- Does the nature of the transition change?

A few lattice studies have addressed such issues in the recent past:

- $N_f = 2$ standard rooted staggered fermions, pion masses down to 200 MeV (M. D., S. Mukherjee, F. Sanfilippo, arXiv:1005.5365)
- $N_f = 2+1$, improved (stout smearing) rooted staggered fermions, physical quark masses (G. S. Bali et al., arXiv:1111.4956)
- $N_f = 4$ and $N_c = 2$ QCD, standard staggered fermions and equal quark charges to avoid rooting (E. -M. Ilgenfritz et al., arXiv:1203.3360; arXiv:1310.7876)
From arXiv:1005.5365, M. D., S. Mukherjee and F. Sanfilippo, unimproved rooted staggered fermions, $N_t = 4$, $L_s = 16$

**Left:** $\langle \bar{\psi} \psi \rangle$ and Pol. loop vs. temperature for various $B$ quanta at $m_\pi \simeq 200$ MeV. $eB \text{ up to } \sim 1 \text{ GeV}^2$.

**Right:** disconnected chiral susceptibility for the same parameters

- Chiral restoration and deconfinement move together as a function of $B$
- An increase of the strength of the transition is observed
- There is a modest increase in $T_c$, of the order of 2% at $|e|B \sim 1 \text{ GeV}^2$.
- Magnetic catalysis is observed at all temperatures
From arXiv:1111.4956 and arXiv:1206.4205, G. Bali et al.: \(N_f = 2+1\), stout smeared rooted staggered fermions, physical quark masses

**Left:** \(T_c\) vs. \(eB\) from chiral susceptibility

**Right:** relative increment of \(\langle \bar{\psi} \psi \rangle\) vs. \(eB\) at various temperatures

- \(T_c\) decreases as a function of \(B\), by about 10-20% at \(|e|B \sim 1\) GeV^2.
- A similar \(T_c\) change is observed from the Polyakov loop
- Magnetic catalysis changes sign at high \(T\)! (inverse catalysis)
- A slight increase in the transition strength is observed
The magnetic susceptibility of strongly interacting matter

Which kind of material is "strongly interacting matter"?
A question strictly related to the equation of state of the system as a function of $B$

- **DIAMAGNETIC?** free energy density $f$ increases with $B$, pressure decreases
- **PARAMAGNETIC?** free energy density decreases with $B$, pressure increases

The question is, in principle, simple and well posed:

We need the magnetization $M = -\partial f / \partial B$ and the magnetic susceptibility $\chi = -\partial^2 f / \partial B^2$ which are in principle perfectly computable equilibrium quantities.

$\chi > 0 \implies$ PARAMAGNETIC $\quad \chi < 0 \implies$ DIAMAGNETIC

PROBLEM: in the usual lattice setup (compact manifold with periodic b.c.), $B$ is quantized and the derivative is not well defined.
That has represented a challenge in early studies.
The idea is to reconstruct directly the $B$-dependent part of the free energy density in place of its derivatives $\Delta f(B, T) = -\frac{T}{V} \log \left( \frac{Z(B, T, V)}{Z(0, T, V)} \right)$.

However, a direct determination of the ratio of partition functions is hardly feasible

$$\frac{Z(B, T, V)}{Z(0, T, V)} = \frac{\int DU e^{-S_G[U]} \det M[U, B]}{\int DU e^{-S_G[U]} \det M[U, 0]} = \left\langle \frac{\det M[U, B]}{\det M[U, 0]} \right\rangle_{B=0}$$

difficulties emerge both in computing the observable and in correctly sampling it.

A standard trick is to rewrite the ratio as the product of intermediate, easily computable ratios of interpolating partition functions (like the ’t Hooft loop), possibly also a continuous interpolation → derivative method (like for the pressure)

$$\log \left( \frac{Z'}{Z} \right) = \log \left( \frac{Z'}{Z_N} \frac{Z_N}{Z_{N-1}} \cdots \frac{Z_2}{Z_1} \frac{Z_1}{Z} \right) = \log \frac{Z}{Z_N} + \cdots + \log \frac{Z_1}{Z} \rightarrow \int_{Z}^{Z'} dx \frac{d \log Z(x)}{dx}$$

NOTICE: Any interpolation is good! Provided the reconstruction is unambiguous
Our idea is to extend the definition of $f(b)$ also to non-integer, unphysical values of $b$, and to obtain physical differences as follows:

$$f(b_2) - f(b_1) = \int_{b_1}^{b_2} \frac{\partial f(b)}{\partial b} \, db,$$

with $b_1$ and $b_2$ integers, computing the integrand on a grid of points.

$\partial f/\partial b$ is not the "magnetization", but just a derivative of the interpolating free energy. As long as the $f(b)$ is differentiable, the procedure is unambiguous.

In practice, our choice for the interpolating $f$ corresponds to the same $U(1)$ field defined above, which for non-integer $b$ describes a uniform field plus a (visible) Dirac string.

On a finite lattice, analyticity is always guaranteed.
example of interpolating magnetic field on a $4 \times 4$ lattice torus
the plaquette in the up-right angle is pierced by the Dirac string
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In practice:

• Consider QCD with fermions in the rooted staggered formulation

\[ Z \equiv \int \mathcal{D}U e^{-S_G} \prod_f \det D^{1/4}[U, m_f, q_f] \]

where the product runs over the different flavors

• The Dirac operator is

\[ D_{i,j}^{(q)} \equiv am\delta_{i,j} + \frac{1}{2} \sum_{\nu=1}^{4} \eta_\nu(i) (u_\nu^{(q)}(i) U_\nu(i) \delta_{i,j-\hat{\nu}} - u_\nu^{*(q)}(i - \hat{\nu}) U_\nu^\dagger(i - \hat{\nu}) \delta_{i,j+\hat{\nu}}) \]

\[ q_u = 2|e|/3 \text{ and } q_{d/s} = -|e|/3. \]

• The derivative of the interpolation can be expressed as

\[ M \equiv -a^4 \frac{\partial f}{\partial b} = \frac{1}{4L_t L_s^3} \sum_f \left\langle \text{tr} \left\{ \frac{\partial D_{m_f,q_f}}{\partial b} D_{m_f,q_f}^{-1} \right\} \right\rangle \]
Renormalization

- $B$-dependent divergences do not cancel when taking the difference $\Delta f \equiv f(B) - f(0)$, and must be properly subtracted.

- We are interested in the magnetic properties of the strongly interacting thermal medium, which may be probed experimentally. Therefore, our prescription is to subtract the vacuum ($T = 0$) contribution

$$\Delta f_R(B, T) = \Delta f(B, T) - \Delta f(B, 0)$$

no further divergences, depending both on $B$ and on $T$, appear

- Divergences are really removed only if the contributions to $f_R$ are evaluated at a fixed value of the lattice spacing.
Effects of QED quenching

- for a linear homogeneous, isotropic medium, the magnetization is proportional to the field (SI units)
  \[\mathcal{M} = \tilde{\chi}B/\mu_0; \quad \mathcal{M} = \chi H; \quad H = B/\mu_0 - \mathcal{M}; \quad \chi = \tilde{\chi}/(1 - \tilde{\chi})\]

- After subtraction of the magnetic field energy in vacuum, one has
  \[\Delta f_R = -\int \mathcal{M} \cdot dB = -\tilde{\chi}/\mu_0 \int B \cdot dB \simeq -\frac{\tilde{\chi}}{2\mu_0}B^2 = -\frac{\tilde{\chi}}{2}(eB)^2\]
  in the small field limit. Last expression defines the susceptibility in natural units.

- \(B\) is the total field felt by the medium. No backreaction from the medium (QED quenching) \(\implies\) it coincides with the external field added to the Dirac operator

- The determination of \(\tilde{\chi}\) is not affected by quenching effects, however, in a real medium, the backreaction would lead to an increase of \(\Delta f_R\) by a factor \(1/(1 - \tilde{\chi})^2\)
Results from $N_f = 2$ unimproved staggered fermions

\[ M \equiv \partial^4 \partial f / \partial b \] on a $T = 0$ and a $T \neq 0$ lattice $a \approx 0.188$ fm

\[ m_\pi \approx 480$ MeV, $T \approx 262$ MeV The lines are third order spline interpolations.

- Oscillating behavior caused by Dirac string becoming more or less visible, two harmonics due to different $u$ and $d$ quark charges

- The area spanned between integer values gives the free energy difference $\Delta f$
• restricting to regions where $a^4 \Delta f(b) \simeq c_2 b^2$ for both $T = 0$ and $T \neq 0$ (linear response region)

$$a^4 (f(b) - f(b - 1)) \equiv \int_{b-1}^{b} M(\tilde{b})d\tilde{b} \simeq c_2 (2b - 1)$$

• Finally: $c_{2R} = c_2(T) - c_2(T = 0)$ and

$$\tilde{\chi} = -\frac{|e|^2 \mu_0 c}{18\hbar \pi^2} L_s^4 c_{2R} ; \quad \hat{\chi} = -\frac{L_s^4 c_{2R}}{18\pi^2}$$
Stability checks

- **Right:** Results change within errors if we refine the grid of points or change the order of the spline integrator.

- **Below** ($M$ and $\int M$): Stability within errors if we change the interpolating free energy: comparison with a "two Dirac strings" interpolation and with adding a constant $A_\mu$ background.

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Results for $N_f = 2$ QCD, standard unimproved staggered fermions

- $\tilde{\chi}$ is small or vanishing below $T_c$, while it steeply rises above deconfinement.
- Numbers indicate strong paramagnetism, one can compare, e.g., $\tilde{\chi} \simeq 2.8 \times 10^{-4}$ for Platinum and $\tilde{\chi} \simeq 3.9 \times 10^{-3}$ for Liquid Oxygen.
- Data show only a mild dependence on the lattice spacing and on the pion mass.
- The fact that the free energy of the deconfined phase decreases with $B$ can account for the fact that $T_c$ decreases with $B$. 

![Graph showing $\tilde{\chi}$ vs. $T$ for different pion masses and lattice spacings.](image_url)
Results confirmed by further studies with improved actions.

  same method, \( N_f = 2 + 1 \) stout staggered fermions at the physical point.

- L. Levkova and C. DeTar, arXiv:1309.1142
  Half-half method: a constant \( B \) for half of the lattice, and a constant \(-B\) for the other half. Zero magnetic flux for every \( B \): no quantization is required. But interface effects at the boundary must be kept under control.
  \( N_f = 2 + 1 \) HISQ fermions

  same idea of thermodynamic integration, but in this case the path goes along the quark mass, in a return trip to the quenched limit. \( N_f = 2 + 1 \) stout staggered fermions

  computation of the magnetization based on pressure anisotropies \( N_f = 2 + 1 \) stout staggered fermions
Collection of results for the magnetic susceptibility of $N_f = 2 + 1$ QCD

- behavior at high $T$ consistent from various determinations
- Quark-Gluon Plasma shows a linear response for a magnetic fields up to $0.2 \text{ GeV}^2$, i.e. those relevant to heavy ion collisions
- situation below $T_c$ still unclear: should be diamagnetic according to HRG (pion gas), but statistical errors still do not allow a definite statement.
The separation of electric charges in the presence of magnetic and topological backgrounds has been one of the original motivation for interest in $B$ effects: (Vilenkin, 1980; Kharzeev, Fukushima, McLerran and Warringa, 2008).

It has been studied on the lattice in various ways:

Full QED+QCD $N_f = 2 + 1$ domain wall study.

Charge separation in presence of an instanton + magnetic field background

P. V. Buividovich et al., PRD 80, 054503 (2009),

Quenched $SU(2)$ study

Electric current fluctuations parallel to the background field are enhanced with respect to transverse ones.

Standard $N_f = 2$ Wilson fermions plus axial chemical potential $\mu_5$ to directly create a net unbalance in chirality (no sign problem related to $\gamma_5$). $m_\pi \sim 400$ MeV and $T > T_c$.

$$j_3 \propto \mu_5 qB$$

as expected, even if overall magnitude lower than predicted by analytic computations (K. Fukushima, D. E. Kharzeev and H. J. Warringa, PRD 78, 074033 (2008)).
G. Bali et al., 1401.4141: measurement of the correlation between electric polarization and topological charge in the presence of a magnetic background

\[
C_f \equiv \frac{\hat{r}_f}{\tau_f} = \frac{\langle q_{top}(x) \cdot \Sigma_{zt}^f(x) \rangle}{\sqrt{\langle q_{top}^2(x) \rangle \langle \Sigma_{xy}^f(x) \rangle}}
\]

\(C_f \sim O(0.1)\) with little \(T\) dependence. Model predictions: \(C_f \sim O(1)\)

Lattice results show a consistent suppression of the effect, with respect to model predictions. However, new lattice studies with chiral fermions would be welcome, to arrive to a definite statement.
Magnetic effects on the gluon sector

Magnetic field interacts with gluons at 1-loop level, but leads to sizable effects.

How does the breaking of an exact symmetry propagate from the e.m. to the gluon sector? One relevant symmetry is CP. The CP-breaking $\theta$ term is absent in QCD: $|\theta| \lesssim 10^{-10}$.

$$-i\theta Q = -i\theta \int d^4 x \, q(x) = -i\theta \int d^4 x \frac{g^2}{64\pi^2} G_{\mu\nu}^a(x)\tilde{G}_{\mu\nu}^a(x)$$

Does a CP-odd background field ($\vec{E} \cdot \vec{B} \neq 0$) induce an effective $\theta$ term?

$$\theta_{\text{eff}} \simeq \chi_{CP} e^2 \vec{E} \cdot \vec{B} + O((\vec{E} \cdot \vec{B})^3)$$

The effect is complementary to the CME, where local CP-breaking propagates back from the gluon to the e.m. sector (charge separation along the magnetic field)

$\chi_{CP}$ is related to the strength of the effective pseudoscalar QED-QCD interaction

$$\chi_{CP} \ q(x) \ e^2 \ \vec{E} \cdot \vec{B} = \kappa \alpha \alpha_s (\vec{E}^a \cdot \vec{B}^a) (\vec{E} \cdot \vec{B})$$

(Elze, Muller, Rafelski, hep-ph/9811372; Asakawa, Majumder, Muller, Phys. Rev. C 81, 064912 (2010).)
Simulations are feasible only if $\vec{E} = i \vec{E}_I$ is purely imaginary! $\Rightarrow \theta_{\text{eff}} = i \theta_{{I\text{eff}}}$. 

Hence we expect an effective $\exp(-\theta_{\text{eff}} Q)$ in the gauge field measure, which will shift the distribution of the topological charge $Q$

$$Q = \int d^4x \, q(x) = \int d^4x \, \frac{g^2}{64\pi^2} G_{\mu\nu}^a(x) \tilde{G}_{\mu\nu}^a(x)$$

**Figures:** MC histories (left) and distribution (right) of $Q$ for different $\vec{E}_I \cdot \vec{B}$ for $N_f = 2$ standard staggered fermions, on a $16^4$ lattice with $a \simeq 0.12$ fm and $m_\pi \simeq 480$ MeV.
Results for $a = 0.3$ fm

$m_\pi \sim 480$ MeV, standard staggered fermions, $16^4$ lattice.

- $\theta_{\text{eff}}$ is odd in $\vec{E} \cdot \vec{B}$ and linear in it for small enough fields, as typical for a system with a linear response. From the slope one can extract $\chi_{CP}$, which turns to be in agreement with phenomenological estimates (a few GeV$^{-4}$).
Is confinement affected by a magnetic background?

– Confinement is a property of gauge fields, which are not coupled to e.m. fields, so one would expect it is not so affected

– However, significant effects on gauge fields are already well known

May we expect an effect also on the static quark-antiquark potential?

\[ V_{Q\bar{Q}}(\vec{r}) = C + \sigma |\vec{r}| + \frac{\alpha}{|\vec{r}|} \]

– \( \sigma \equiv \text{String Tension} \)

– \( \alpha \equiv \text{Coulomb Parameter} \)

The magnetic fields breaks rotational invariance, does the breaking propagates to pure gauge quantities? The breaking is already seen in plaquette expectations values, might be visible also in static potential.
$V_{Q\bar{Q}}$ can be extracted by computing the Wilson Loop

\[ W(\vec{R}, T) \]

A rectangular $R \times T$ loop built up of link variables $U_\mu(n)$.

$$aV_{Q\bar{Q}}(a\vec{n}) = \lim_{n_t \to \infty} \log \left( \frac{W(a\vec{n}, a(n_t))}{W(a\vec{n}, a(n_t+1))} \right)$$

- creation of a quark-antiquark pair at distance $\vec{R}$
- imaginary time propagation for an interval of time $T$
- annihilation of the pair.

\[ \langle W(\vec{R}, T) \rangle \simeq C \exp \left( -TV_{Q\bar{Q}}(\vec{R}) \right) \]
An anisotropy clearly emerges, the potential is steeper in the directions orthogonal to the magnetic field.

Results for $a \sim 0.125 \text{ fm}$ on a $40^4$ lattice.

Remarkably, if one averages the Wilson loop over all directions, the potential is unchanged.
The string tension (left) increases in the orthogonal directions and decreases in the longitudinal directions.

The Coulomb term (right) shows an opposite behavior.

– Does that have consequences on the phenomenology of heavy ion collisions?
– Masses of heavy mesons in a magnetic field?
– Could the effect add to already predicted anisotropies in heavy meson productions in non-central heavy ion collisions?